Loewner Evolution

Some preliminaries:

1) Subordination:

Let $f, g \in A(\mathbb{D})$, g - conformal. We say that f is subordinate to g, $f \prec g$ if $\exists\ \varphi: \mathbb{D} \to \mathbb{D}$ - analytic, $\varphi(0)=0$, $f = g \circ \varphi$.

Equivalently: $f(\mathbb{D}) \subset g(\mathbb{D})$, $f(0) = g(0)$. Pf: $\varphi = g^{-1} \circ f$.

Consequences of subordination:

a) $\{f(z) : |z| < r\} \subset \{g(z) : |z| < r\}$ (since $|\varphi(z)| \le |z|$).

b) $|f'(0)| \le |g'(0)|$ ($|\varphi'(0)| \le 1$).

c) $\max (1-|z|^2)|f'(z)| \le \max (1-|z|^2)|g'(z)|$ (By Schwarz: $\frac{|\varphi'(z)|}{1-|\varphi(z)|^2} \le \frac{1}{1-|z|^2}$).

2) Class $\mathcal{P}$

Def

For $p \in A(\mathbb{D})$. $p \in \mathcal{P} \Leftrightarrow p \prec \frac{1+z}{1-z} \Leftrightarrow \operatorname{Re} p > 0$, $p(0) = 1$.

1° 1) $\frac{1-|z|}{1+|z|} \le |p(z)| \le \frac{1+|z|}{1-|z|}$

2) $|p'(z)| \le \frac{2}{1-|z|^2}$

2° $\mathcal{P}$ is locally bounded $\Rightarrow$ normal.

Herglotz representation: $p \in \mathcal{P} \Rightarrow$
$\exists\ \mu$ - probability measure on $\mathbb{T}$:

$$p(z) = \int \frac{\zeta + z}{\zeta - z}\, d\mu(\zeta)$$

$\operatorname{supp} \mu = \operatorname{clos}\{\zeta \in \mathbb{T} : \lim_{r \to 1^-} \operatorname{Re} p(r\zeta) \neq 0\}$.

Pf Write the Poisson representation for positive $\operatorname{Re} p$, take conjugate ∎

Classical Löwner Chain (or Radial Löwner Chain)

Def. (f_t) - collection of conformal mappings from $\mathbb{D}$, $t \in [0, \infty)$, such that

1) $t_1 < t_2 \Rightarrow f_{t_1} \succ f_{t_2}$

2) $f_t(z)$ is continuous in t, uniformly on compacts.

3) $f_0(z) = z$, $\lim_{t \to \infty} f_t(z) = 0$

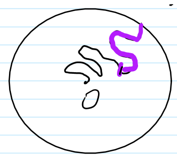
is called Löwner chain (non-normalized)

Equivalent geometric def (in terms of $\Omega_t := f_t(\mathbb{D})$).

1) $0 \in \Omega_t \quad \forall t$.
2) $t_1 < t_2 \Rightarrow \Omega_{t_1} \supset \Omega_{t_2}$.
3) $\Omega_0 = \mathbb{D}, \quad 0 \notin \operatorname{Int} \bigcap_{t \to \infty} \Omega_t$.
4) $t \to \Omega_t$ is Carathéodory continuous w.r.t. 0.

$K_t := \mathbb{D} \setminus \Omega_t$ – growing subsets, hulls.

Example. Slit domains:



$\gamma : [0,\infty] \to \overline{\mathbb{D}}$,
$\gamma(\infty) = 0$
$\gamma(0) \in \mathbb{T}$
$\gamma[0,\infty] \subset \mathbb{D}$.

$\Omega_t := \mathbb{D} \setminus \gamma[0,t]$.
$K_t = \gamma[0,t]$.

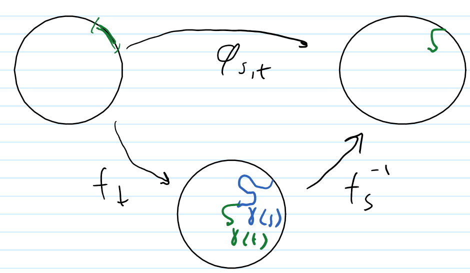
Can be self-touching.

Ω_t – component of 0 of $\mathbb{D} \setminus \gamma[0,t]$.
$K_t = \mathbb{D} \setminus \Omega_t$

Observe: $a(t) := f_t'(0)$ – continuous, $a(0) = 1$, $\lim_{t \to \infty} a(t) = 0$; monotone (by 1) continuous (by 2)

Can reparametrize so that $a(t) = e^{-t}$

Def. Normalized L.C. (or simply Loewner Chain): 1) – 3) + 4) $f_t'(0) = e^{-t}$.



For $s < t$, define $\varphi_{s,t}(z) = f_s^{-1} \circ f_t(z) : \mathbb{D} \to \mathbb{D}$ – conformal,
$\varphi_{s,t}'(0) = e^{s-t}$ Notation: $\varphi(z,s,t) := \varphi_{s,t}(z)$

Chain relation: $s \le t \le \tau$. Then
$\varphi(z,s,\tau) = \varphi(\varphi(z,t,\tau), s, t)$.
$(f_s^{-1} \circ f_\tau = (f_s^{-1} \circ f_t) \circ (f_t^{-1} \circ f_\tau))$.

Key observation.

$$p_{s,t}(z) = p(z,s,t) := \frac{1+e^{(s-t)}}{1-e^{(s-t)}} \; \frac{z - \varphi_{s,t}(z)}{z + \varphi_{s,t}(z)} \in \mathcal{P}. \quad s<t.$$

So $\underline{p_{s,t}(z) = \int \frac{\zeta+z}{\zeta-z}\, d\mu_{s,t}(\zeta)}$, $\underline{\operatorname{supp} \mu_{s,t}(z) = \operatorname{Clos}\{\zeta \in \mathbb{D} : \overline{\lim_{r \to 1-}} |\varphi_{s,t}(r\zeta)| \ne 1\}}$.

Key trick Write

$$\frac{f_t(z) - f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \cdot \frac{\varphi_{s,t}(z) - z}{t-s} =$$

$f_s(\varphi_{s,t}(z)) - f_s(z)$ [illegible]

$$\frac{f_t(z)-f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{t-s} = \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \cdot \frac{\varphi_{s,t}(z)-z}{t-s} =$$

$$\frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \cdot \underbrace{\frac{z-\varphi_{s,t}(z)}{z+\varphi_{s,t}(z)} \frac{1+e^{s-t}}{1-e^{s-t}}}_{p_{s,t}(z)} \frac{(e^{s-t}-1)(z+\varphi_{s,t}(z))}{(1+e^{s-t})(t-s)} \quad (*)$$

$$p_{s,t}(z) = \int \frac{z+\zeta}{z-\zeta} \, d\mu_{s,t}$$

Formally, if we let $t \to s$, we get

$$\boxed{\frac{\partial f_t(z)}{\partial t} = -f_t'(z)\, z \int \frac{\zeta+z}{\zeta-z} \, d\mu_t}$$ – Löwner equation.

Now, let us prove it!

Lemma. $(f_t(z))$ – L.C. Then

1) $|f_t(z) - f_s(z)| \le \frac{8|z|}{(1-|z|)^4} (e^{-t} - e^{-s})$

2) $|\varphi_{t,u}(z) - \varphi_{s,u}(z)| \le \frac{2|z|}{1-|z|^2} (1 - e^{s-t})$ $\quad t > s > u.$

Pf $\quad |p_{s,t}(z)| \le \frac{1+|z|}{1-|z|}$, so

$$|\varphi_{s,t}(z) - z| \le |\varphi_{s,t}(z) + z| \frac{1-e^{s-t}}{1+e^{s-t}} \frac{1+|z|}{1-|z|} \le 2|z| \frac{1+|z|}{1-|z|} (1-e^{s-t})$$

Now $|f(z,t) - f(z,s)| = \left| \int_z^{\varphi_{s,t}(z)} f'(\xi, s) d\xi \right|$. Use distortion Thm to estimate $|f'(z,\xi)|$. Same for φ. ∎

So $t \to f_t(z)$ – Lipshitz $\Rightarrow$ a.e. differentiable.

$\Leftarrow (*)$ again. $\lim_{t \to s}$ LHS exists a.e. t for all z (do it for dense countable set, use uniform continuity on compacts).

Let $t > s$

$$\lim_{t \to s} \frac{f_s(\varphi_{s,t}(z)) - f_s(z)}{\varphi_{s,t}(z) - z} \to f_s'(z), \text{ by Caratheodory continuity } (\varphi_{s,t}(z) \to z \text{ unif. on compacts})$$

$$\lim_{t \to s} \frac{e^{(s-t)}-1}{1+e^{s-t}} \frac{(z+\varphi_{s,t}(z))}{t-s} = -z, \text{ again since } \varphi_{s,t}(z) \to z.$$

so $\exists \lim_{t \to s} p_{s,t}(z) =: p_s(z)$ a.e. s.

Same for $t < s$ – again $\varphi_{t,s}(z) \to z$ as $t \to s$.

We just proved half of Löwner Theorem.

Thm (f_t) is a normalized Löwner Chain iff

1) $\forall t$ $f_t \in A(\mathbb{D})$, $t \to f_t(z)$ – absolutely continuous $\forall z \in \mathbb{D}$.

2) $\exists (p_t)$ – measurable in t family of functions from $\mathcal{P}$, such that a.e. t, $\forall z \in \mathbb{D}$, $\frac{\partial f_t}{\partial t} = -z \frac{\partial f_t(z)}{\partial z} p_t(z)$

2) $\exists (p_t)$ - measurable in t family of functions from $\mathcal{P}$, such that a.e.t, $\forall z \in \mathbb{D}$, $\frac{\partial f_t}{\partial t} = -z \frac{\partial f_t(z)}{\partial z} p_t(z)$.

Let us now prove the other direction.

First, let us consider $\varphi_{s,t}(z)$.

Observe:

$0 = \frac{\partial}{\partial s} f_t(z) = \frac{\partial}{\partial s} f(\varphi(z,s,t), s) = \frac{\partial}{\partial s} f(\varphi, s) + f'(\varphi, s) \cdot \frac{\partial}{\partial s} \varphi(z,s,t)$

On the other hand, by Löwner at the point $\varphi = \varphi(z,s,t)$:

$\frac{\partial}{\partial s} f(\varphi, s) + \varphi p(\varphi, s) f'(\varphi, s) = 0$.

Then, since $f'(\varphi, s) \neq 0$, $\boxed{\frac{\partial}{\partial s} \varphi(z,s,t) = \varphi(z,s,t)\, p(\varphi(z,s,t), s).}$

Remark

$\frac{\partial}{\partial s}(f_s^{-1} \circ f_t) = (f_s^{-1} \circ f_t)\, p_s(f_s^{-1} \circ f_t)$

$g_s := f_s^{-1}$ - inverse $\quad \frac{\partial}{\partial s} g_s = g_s\, p_s(g_s) \quad \forall z \in \Omega_s$.

So if $w(s) := \varphi(z,s,t)$, we have $\frac{dw}{ds} = w\, p_s(w)$, defined for $s \leq t$, with b.c. $w(t) = \varphi(z,t,t) = z$ $(\ast\ast)$.

Let us study $(\ast\ast)$.

Pavel Kufarev (1909-1968)

Thm. (Löwner - Kufarev).

For (p_s) as above, $\frac{dw}{ds} = w\, p_s(w)$ a.e. s has unique solution $w^{t,z}(s)$ for $s \in [0,t]$ with $w^{t,z}(t) = z$. The map $z \to \varphi_{s,t}(z) := w^{t,z}(s)$ is univalent.

The maps $(\varphi_{0,t}(z))_t$ form normalized Löwner Chain.

How Löwner-Kufarev implies other direction of Löwner Thm.

Let (f_s) satisfy 1) and 2) of Löwner (a.e., but fine, by absolute continuity)

Then $\frac{\partial}{\partial s} f_s(w^{t,z}(s)) \overset{\text{chain rule}}{=} f_s'(w) \frac{\partial w^{t,z}(s)}{\partial s} + \frac{\partial f_s(w)}{\partial s} \overset{\text{Löwner}}{=} 0$.

So $f_s(w^{t,z}(s))$ is constant in s, equal to $f_t(z)$.

So $f_t(z) = f_s(\varphi_{s,t}(z))$. In particular, $f_t(z) = f_0(\varphi_{0,t}(z)) = \varphi_{0,t}(z)$ - Löwner Chain

Proof of Löwner-Kufarev.

Use Pickard-Lindelöf iteration.

For now, fix z, $r := |z|$.
Rewrite as integral equation:
$w(s) := z\exp\left(-\int_s^t p(w(\tau),\tau)\,d\tau\right)$
Let $w_0(s) := 0$
$w_{n+1}(s) := z\exp\left(-\int_s^t p(w_n(\tau),\tau)\,d\tau\right)$. Observe: $|w_n(s)| \le |z|$.

Note: $p \in \mathcal{P}$, so $|p'(\zeta,\tau)| \le \frac{2}{(1-|\zeta|)^2}$; also, for $\substack{\operatorname{Re} a<0 \\ \operatorname{Re} b<0}$ $|e^a - e^b| \le |a-b|$

so $|w_{n+1}(s) - w_n(s)| \le \int_s^t |p(w_n(\tau),\tau) - p(w_{n-1}(\tau),\tau)|\,d\tau \le \frac{2}{(1-r)^2}\int_s^t |w_n(\tau) - w_{n-1}(\tau)|\,d\tau$

Thus, by induction, $|w_{n+1}(s) - w_n(s)| \le \frac{2^n}{(1-r)^{2n}}\frac{(t-s)}{n!}$

Thus $\lim_{n\to\infty} w_n(s)$ exists, uniformly on compacts.
Also, for each fixed $s \le t$, $z \to w_n(s)$ is analytic in z. So is w.
By Lebesgue bounded convergence
$w(s) = z\exp\left(-\int_s^t p(w,\tau)\,d\tau\right)$. Thus satisfies $(**)$.
Observe: $\frac{d|w|^2}{ds} = 2|w|^2 \operatorname{Re} p_s(w)$ so $|w|$ is increasing in s.
Now note that $\left|\frac{d}{ds}(w(s) - v(s))\right| = |w p(w,s) - v p(v,s)| \le k(r)|w-v|$, where k comes from distortion for $\mathcal{P}$. Thus if $w(t) = v(t)$ then for all $s<t$ $w(s) = v(s)$. Thus, we have the uniqueness, by Gronwall.
By uniqueness, we have the chain relation $\varphi(z,s,\tau) = \varphi(\varphi(z,t,\tau),s,t)$, $s \le t \le T$.
(both satisfy equation in s, and for $s=t$ LHS $= \varphi(z,t,\tau) =$ RHS
$= \varphi(\varphi(z,t,\tau),t,t) = \varphi(z,t,\tau)$).
By the same uniqueness: $\varphi(z_1,s,t) = \varphi(z_2,s,t) \Rightarrow \varphi(z_1,t,t) = \varphi(z_2,t,t) \Rightarrow$
$z_1 = z_2$, so φ is conformal

Observe, that by chain relation with $s=0$
$\varphi(z,0,\tau) = \varphi(\varphi(z,t,\tau),0,t)$. So if $f_t := \varphi(z,0,t)$, we get
$f_\tau = f_t(\varphi(z,t,\tau))$ for $\tau > t$. So $f_t \succ f_\tau$.
Also, by the same relation, $\tau \to f_\tau$ is uniformly continuous on compacts.

$0 = \frac{\partial}{\partial t} f_\tau(t) = \frac{\partial}{\partial t} f_t(\varphi(z,t,\tau)) = \frac{\partial}{\partial t} f_t(\varphi(z,t,\tau)) + f_t'(\varphi(z,t,\tau))\frac{\partial}{\partial t}\varphi(z,t,\tau) = \frac{\partial}{\partial t} f_t(\varphi(z,t,\tau)) +$
$f_t'(\varphi(z,t,\tau))\varphi(z,t,\tau)\,P(\varphi(z,t,\tau),t)$. Take $\tau = t$ to get
$\frac{\partial}{\partial t} f_t(z) = -f_t'(z)\,z\,P_t(z)$ ■